Relativity Terms Becoming Important in GNSS

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Relativity Terms in GPS Not Yet in Use

- Signal Propagation Delay Caused by Earth's Gravitational Field up to 2 mm = 6 ps on Earth
- Signal Propagation Delay for Cross-Link Caused by Earth's Gravitational Field up to 35 ps

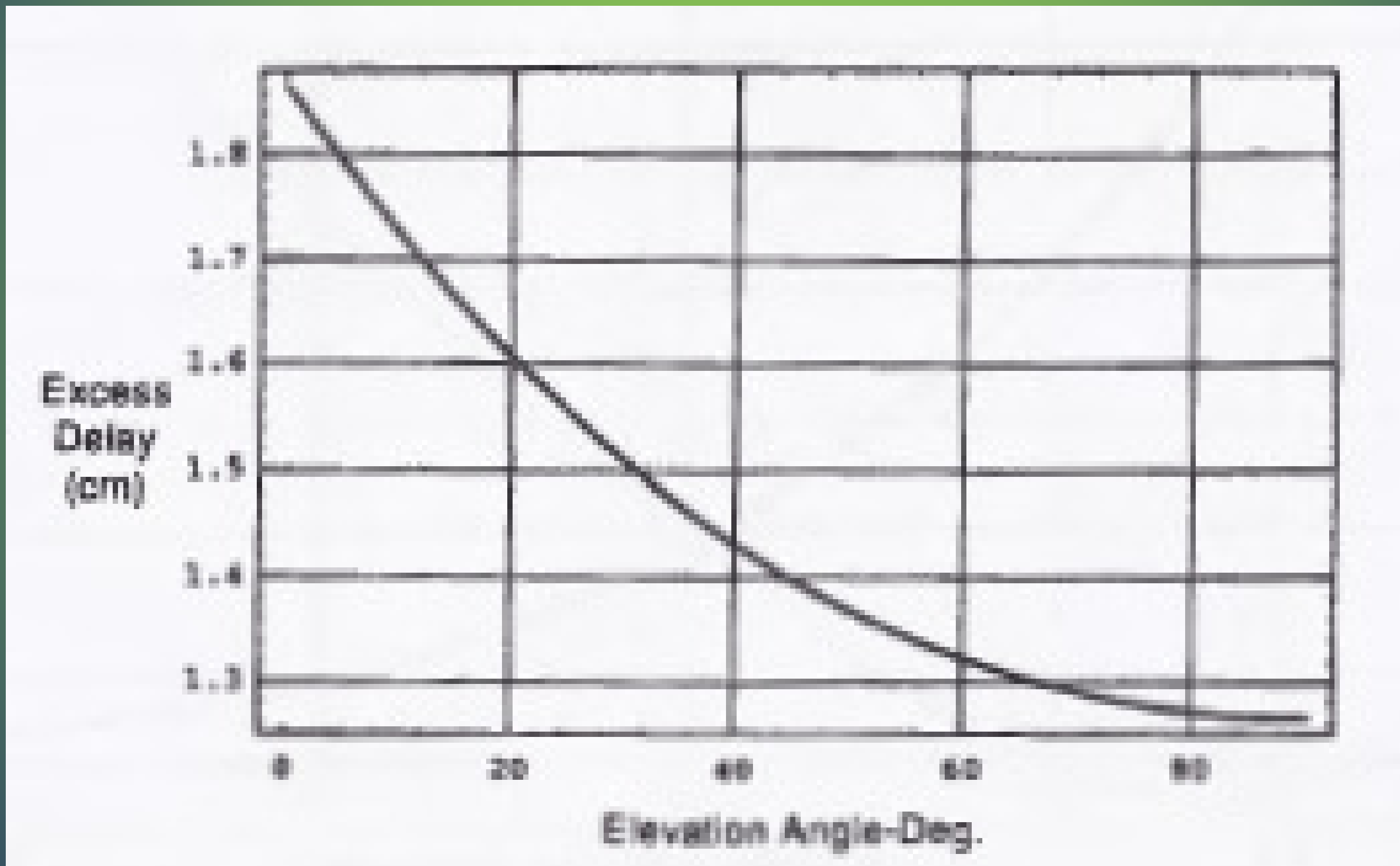
- Clock Rates Changed by the Luni-Solar Tidal Potential perhaps 4 x 10^{15} with 6 hour period in satellite clocks

Signal Propagation Delay Caused by Earth's Gravitational Field

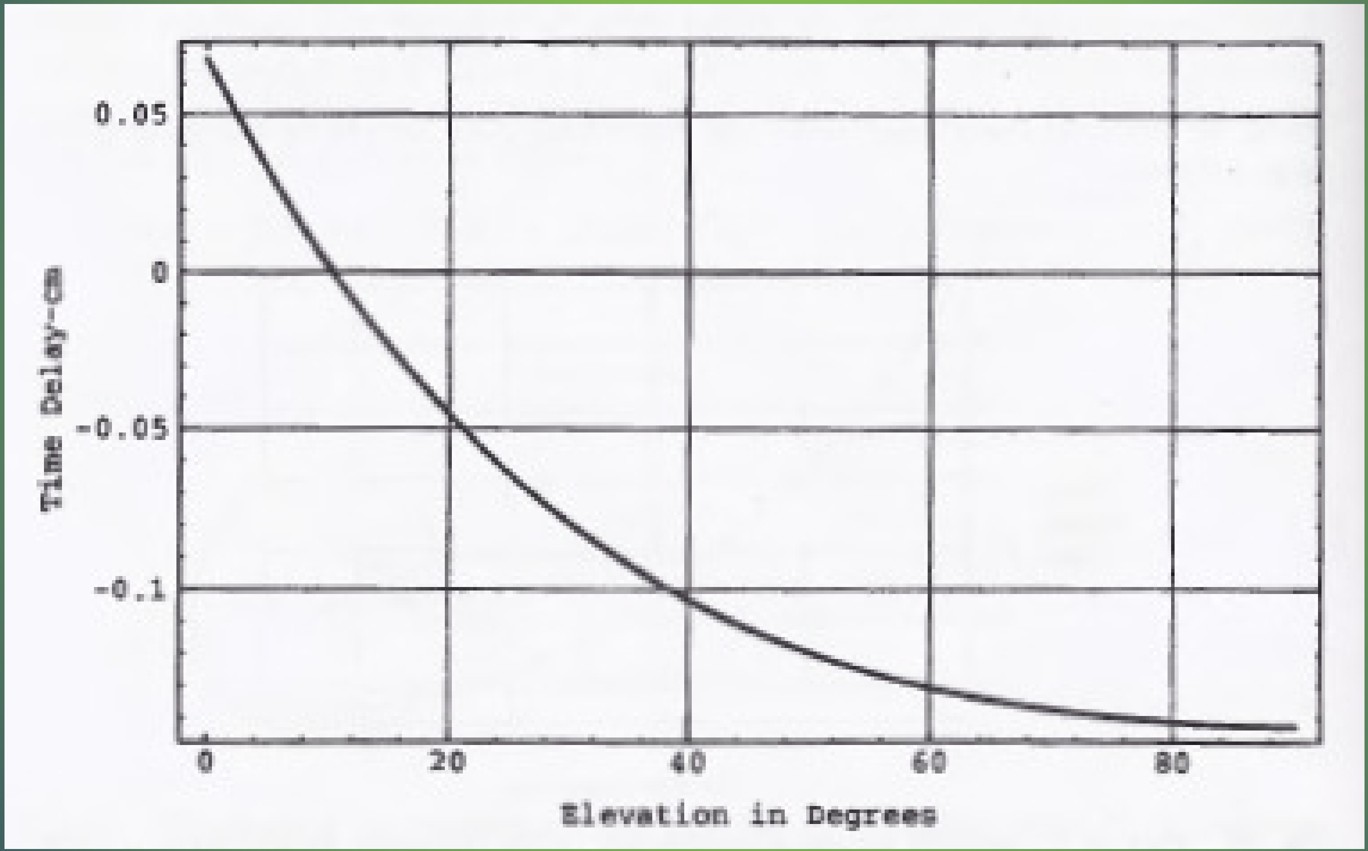
- Coordinate speed of light is generally slower than *c* due to Earth's gravitational field
- The length of a light path is the coordinate length *l* plus the so-called Shapiro delay

$$path\,length = c \bullet \Delta t = l + \frac{2GM_E}{c^2}\int_{path}\frac{dl}{r}$$

Excess Path Length as Measured by a Clock at Infinity

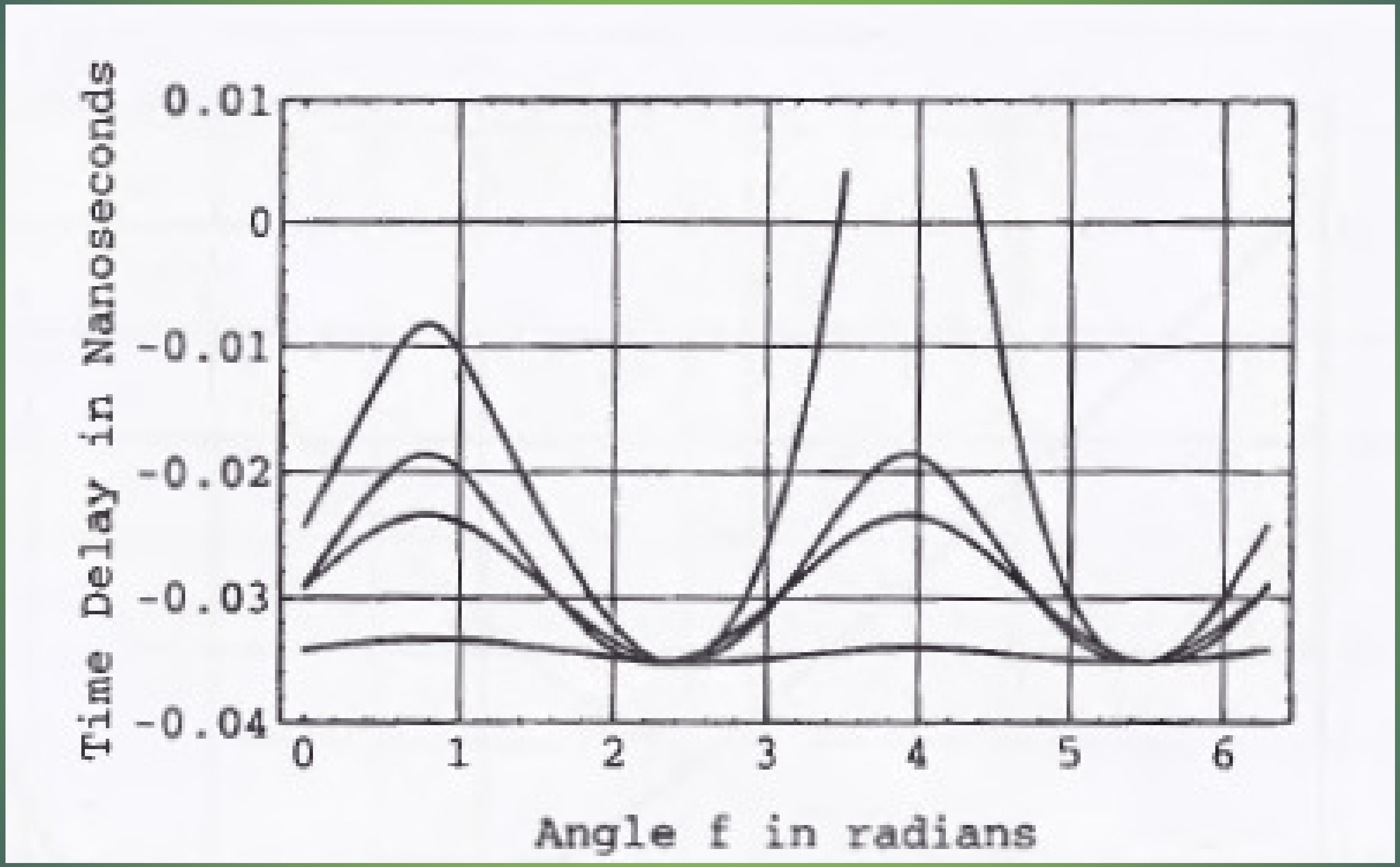


Excess Path as Measured by a Clock on the Geoid, a “TAI” Clock



Peak-Peak about 2 mm

Excess Path Delay for Cross-Link



Horizontal axis is angle in radians of sat#1 above equatorial plane.
Different curves are angles in 60 deg steps of sat#2 above equatorial plane.

Relativity and Clock Rates

- Clocks run slower deeper in gravi-potential
- Clocks run slower as they go faster

Relativity and Clock Rates

- Proper time τ vs. coordinate time t

$$d\tau^2 = \left(1 + \frac{2(\Phi - \Phi_0)}{c^2}\right)dt^2 - \frac{dx^2 + dy^2 + dz^2}{c^2} = \left(1 + \frac{2(\Phi - \Phi_0)}{c^2} - \frac{v^2}{c^2}\right)dt^2;$$

$$d\tau \simeq \left(1 + \frac{\Phi - \Phi_0}{c^2} - \frac{1}{2}\frac{v^2}{c^2}\right)dt.$$

Where is Φ the gravitational potential, Φ_0 the potential on the geoid, this applies for clocks moving with velocity v relative to the geoid slowly relative to light speed.

Relativity and Clock Rates

- Fractional frequency offset

$$\left.\frac{\Delta f}{f}\right|_{gravipotential} = \frac{\Phi - \Phi_0}{c^2}$$

$$\left.\frac{\Delta f}{f}\right|_{speed} = -\frac{v^2}{2c^2}$$

Expansion of Gravitational Potential

M is the mass of the Moon or Sun, *R* is the vector to Earth's center, *r* is the vector from Earth to satellite, expanding in *r*

$$\Phi = -\frac{GM}{\left|\mathbf{R}+\mathbf{r}\right|} = -\frac{GM}{R} + \frac{GM\ \mathbf{R}\cdot\mathbf{r}}{R^3} - GM\,\frac{3\left(\mathbf{R}\cdot\mathbf{r}\right)^2 - R^2 r^2}{2R^5} + \ldots$$

The first two terms have no net effect, the third term is the tidal potential: decreases as the cube of the distance from the sun or moon, grows as the square of the distance from Earth, and depends on the angle between *R* and *r*.

Three Separate Contributions to Tidal Effect on Clocks

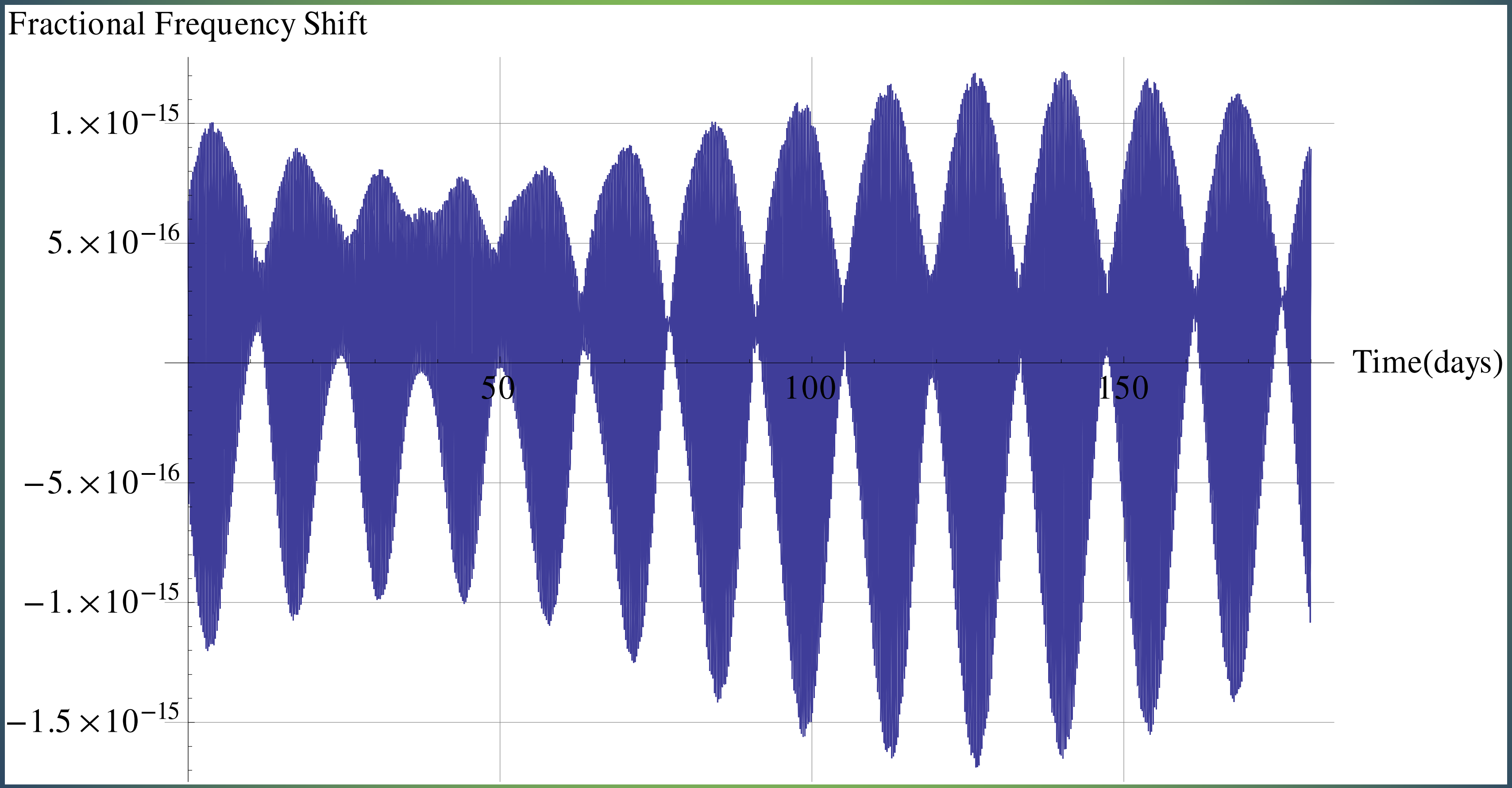
1. Fractional frequency shift due to the tidal potential itself
2. The tidal perturbation causes a change in radius that contributes to the fractional frequency shift
3. There is also a change in satellite speed

Many Distinct Rates of Change Occur in the Fractional Frequency Shift

Inclination of the following with respect to each other, and the position of bodies in their orbits, determine the summation of effects:

- Earth's equatorial plane
- The ecliptic plane
- The plane of the moon's orbit
- The GPS satellite's plane

Fractional Frequency Shift for a Nominal GPS (Circular) Orbit Due to Sun and Moon



Peak-Peak (not shown) is 3.2×10^{-15}

Total Effect and Future Work

- Work not completed for full analysis
- Preliminary estimates indicate that eccentricities can increase the effect about 10%
- Effect grows as the square of the radius of orbit hence the effect Galileo will be about 25% larger
- Need to group hundreds of terms in ways that could be observed
- Specific calculations for a GPS satellite with a very stable clock will verify results

Summary

- Accounting for the Shapiro delay by receivers locked to GPS could contribute 2 mm = 6 ps to range on Earth
- Signal Propagation Delay for Cross-Link Caused by Earth's Gravitational Field up to 35 ps

- Accounting for variations in GPS clocks due to Solar and Lunar tidal potentials could account for 4×10^{-15} with a period of 6 hours, plus other effects
- These effects could be 25% larger in Galileo satellites